Stellations, compounds, periodicity and the exponential scale

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Abstract

The stellations of the octahedron, the dodecahedron, some of the icosahedron including the final stellation, and of the cubeoctahedron, are described with periodic GD functions.

1 Introduction

There are beautiful descriptions of these geometric solids using paper sloyd art and Wenninger’s Polyhedron Models is already a classic(ref 1). We have here followed Wenninger’s strategy, which also contains descriptions and references to Coxeter’s pioneering work.

Easily available are recent works on the web just searching for polyhedra and stellations you find plenty of paper sloyd, and ordinary plotting. And also more Wenninger work up to now at http://employees.csbsju.edu/mwenninger/photosset2/phot-mw2.html

Using the GD function and the exponential scale we describe stellations and some compound polyhedra below. The polyhedra are the octahedron, the dodecahedron, the icosahedron and the cubeoctahedron.

2 The method and the octahedron

The GD function is a peaked function and as such used to describe not only heat and diffusion gradients, but also wave packets, and soliton motion. Peaked functions were also found in the mathematics and physics of the dispersion forces(ref 2), and used to describe why ice floats on water (ref 3). Of the same reason we use it to describe stellated polyhedra – and the mathematics is simple. The roots of the understanding of these operations lie in the understanding of periodicity.

We recently described finite periodicity (ref 4) and give only a simple example here.

First we use equation 2.1 to describe two spheres in space in fig 2.1. This is handmade periodicity, and there are just two spheres, no more no less. This is because there is a sphere in each term in eq 2.1, and a sphere has finite extension in space.

\[ e^{\pi(x^2+y^2+z^2)} + e^{\pi((x-2)^2+y^2+z^2)} = .9 \] 2.1
In equation 2.2 there are three GD functions that each has extension in space and give a hyperbolic octahedron in fig 2.2a. Which then must contain peaked functions in three directions of space.

\[ e^{-x^2} + e^{-y^2} + e^{-z^2} = 2 \]  

We add two of these side by side in space after eq 2.3.

\[ e^{-x^2} + e^{-y^2} + e^{-z^2} + e^{-(x-4)^2} + e^{-(y-4)^2} + e^{-(z-4)^2} = 2 \]  

And obtain eight octahedra in space as seen in fig 2b.

This was an example of true periodicity, which is finite contrary to cosine.

We are now ready for the stellations and start with the octahedron and use the square of the planes as the structure is centric. We go up with the exponentials in eq 2.4 to get sharp edges, as we are doing polyhedra. The picture is in fig 2.3a.

\[ 10^{(x+y+z)^10} + 10^{(x+y+z)^10} + 10^{(x+y+z)^10} + 10^{(x+y+z)^10} = 10^6 \]  

2.4
And of reasons said above we switch to a GD type function in eq 2.5 to get the stellation, as in fig 2.3b, which is the only stellation there is of the octahedron. It is also called stella octangula, after Kepler, and is also a compound of two tetrahedra.

\[10^{(x+y+z)^{10}} + 10^{(x+y+z)^{10}} + 10^{(x+y+z)^{10}} = 2.3\]

**3 The dodecahedron and its stellations**

There are three stellations of the dodecahedron and we give them here. We start with the dodecahedron itself in eq 3.1 and show it in fig 3.1a.

\[20^{(x+y)^{20}} + 20^{(x+y)^{20}} + 20^{(x+y)^{20}} + 20^{(x+y)^{20}} = 10^6\]

Using GD style we get the small stellated dodecahedron from eq 3.2 and shown in fig 3.1b.

\[20^{(x+y)^{20}} + 20^{(x+y)^{20}} + 20^{(x+y)^{20}} + 20^{(x+y)^{20}} = 4.1\]
Change of constant to 3.3 gives the great dodecahedron in fig 3.2a

And another change of constant to 2.5 gives the great stellated dodecahedron in fig 3.2b, another Kepler polyhedron of great beauty, and that also is on the cover of Wenninger’s famous book.

4 The icosahedron and some of its stellations

First the icosahedron itself after equation 4.1, which is shown in fig 4.1a.
Using GD style in eq 4.2 we get the first stellation of the icosahedron fig 4.1b for a constant of 5, the second stellation for a constant of 4.5 in fig 4.2a, the great icosahedron for a constant of 3.5 in fig 4.2b.

\[
\begin{align*}
&10(\overline{x+y+z})^{10} + 10(\overline{x+y+z})^{10} + 10(\overline{x+y+z})^{10} + 10(\overline{x+y+z})^{10} + \\
&10(x+y^{2})^{10} + 10(x+y^{2})^{10} + 10(z+y^{2})^{10} + 10(z+y^{2})^{10} + 10(y+z^{2})^{10} + 10(y+z^{2})^{10} \\
= &10^6
\end{align*}
\]

And using a constant of 2 we get the final stellation of the icosahedron. In this formidable periodicity in space there are 60 pikes, which is easy to find out from fig 4.3.
5. The cubeoctahedron and its stellations

First the cubeoctahedron itself as in eq 5.1, and in fig 5.1a.

\[10(x+y+z)^2 + 10(x+y)^2 + 10(x+y+z)^2 + 10(x+y+z)^2 +
10(2x)^2 + 10(2y)^2 + 10(2z)^2 = 10^6\]

Going to the GD style we augment the exponent to get sharp edges. For a constant of 5.3 we have the first stellation, which also is a compound as is easy to see, in fig 5.1b.

\[30(x+y+z)^30 + 30(x+y)^30 + 30(x+y+z)^30 + 30(x+y+z)^30 +
30(2x)^30 + 30(2y)^30 + 30(2z)^30 = 5.3\]

The second and third stellations of the cubeoctahedron is shown in fig 5.2a and b for constants of 4 and 3.1.
Fig 5.1a The cubeoctahedron  
5.1b First stellation of the cubeoctahedron.

Fig 5.2a Second stellation of the cubeoctahedron.  
5.2b Third stellation of the cubeoctahedron.

References


2 Mahanty, J.; Ninham, B.W. DISPERSION FORCES, Academic Press (1976),
